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# A New Method Of Determining Orbit Lifetime Probabilities For Use In Planetary Protection Analysis

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# A NEW METHOD OF DETERMINING ORBIT LIFETIME PROBABILITIES FOR USE IN PLANETARY PROTECTION ANALYSIS*

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A new statistical method was derived to model the long-term behavior of the Martian atmosphere and its effect on a satellite's orbital lifetime. It was successfully used to lower the requirement for the initial quarantine orbit for the Mars Global Surveyor mission to 427 km from the 450 km dictated using conventional methods. A detailed description of the general method will be presented along with how it fits in with the general propagation models used for mission design. The results for MGS will be presented for a variety of input parameters and modeling assumptions. Future applications include other missions and problems in completely different fields, such as species extinction in ecology.

## INTRODUCTION

In order to fulfill the planetary protection requirements for the Mars Global Surveyor (MGS) mission, a capability to predict the probability of various orbital lifetimes was needed. Orbital lifetime depends on drag which is determined by satellite unique drag parameters and the atmospheric density encountered in the orbit. This paper presents a new method of solving the inherent problem of modeling the long-term behavior of the Martian atmosphere. The simple case of one 1 l-year solar cycle will be discussed as an introduction to the more complicated situation of many solar cycles. In particular the relationship between solar flux and atmospheric density will be established.

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The nominal solar flux incident upon Mars had been previously modeled as a combination of three sinusoidal functions. Namely, the dominant 1 l-year term, a smaller 687-day term due to the variation in the Sun-Mars distance caused by the orbital eccentricity and an even smaller 578-day term due to higher order effects. The basis of this study is the statistical nature of the variations about the nominal behavior of the n-year term. Prediction curves indicate the larger variations from nominal are at the solar maximums. However, due to the structure of the software used in this study, the variation was assumed to be a constant (higher) offset throughout the entire 1 l-year cycle. Note that this adds some small amount of conservatism to the study since the offset densities used at solar minimums were thus higher than their actual expected values. The impact is small however, because of the much larger contribution to the drag during the high part of the solar cycle.

The mathematical relationship between the density at a given height and the solar flux that was used was :

$$\rho/\rho_o = 10^{(A_1 \sin(t-\tau_1)/T_1 + 0.3s + A_2 \sin(t-\tau_2)/T_2 + A_3 \sin(t-\tau_3)/T_3)}$$

where the amplitudes (A's), epochs ($\tau$'s) and periods (T's) of the sinusoidal terms are given in Table 1. The density $\rho_o$ corresponds to an average solar flux. The variable s represents the amount of offset that the solar flux has away from its nominal periodic behavior. It is assumed to be a normally distributed variable and is standardized to have the units of sigma's (standard deviations). If $\rho_m$ is defined as the nominal density including the sinusoidal variations, the above equation simplifies to:

$$\rho/\rho_m = 10^{0.3s} \quad \text{or} \quad \log(\rho/\rho_m) = 0.3s$$

The coefficient 0.3 comes from the empirical work by Yen[1]. Thus, given a value of s, it was simple to calculate the scale factor $\rho/\rho_m$ and then use it as an input variable to the computer programs which internally contain the sinusoidal variations.

Table 1
SINUSOIDAL. VARIATIONS IN SOLAR FLUX

| Period (days) | Amplitude | Epoch (MJD) |
|---|---|---|
| 3905.3756 | 0.7085176 | 2451178.0 |
| 687.5263 | -0.5695922 | 2447575.4 |
| 578.4599 | -0.0664183 | 2447419.8 |

MULTIPLE SOLAR CYCLES

Using the probability distribution of one solar cycle described above to equal the total probability for multiple cycles (the "uninomial method") will be described first. This is followed by the description of the binomial method which uses the product of probabilities. The excessive conservatism in using the common binomial method leads to the trinomial method.

The reason for using the term "uninomial method" will be become more apparent later. It implies using the same density offset for all cycles considered. Application to the problem at hand equates to finding the largest single value of density variation for which the satellite will remain in orbit. This density variation has a corresponding probability value. Although using this probability value as the total value for the n cycles does not agree with the cycles being independent, the somewhat surprising result is that the uninomial value is quite close to the true value estimated by the trinomial method. In particular it is much closer than the result using the binomial method (for example: 94.7%, 80.5%, 95.7% for uni-, bi- and trinomial methods).

However, the binomial method is the standard way to treat the problem. Again one value of probability (P) is picked such that integrating the orbit over n cycles at the corresponding value of density does not cause a crash. In this case, the product $P^n$ is equal to the total probability. Mathematically this can be represented by considering the probability of a variable $\alpha$ being either less than (or equal to) a value y or greater than y. The sum of these two possibilities is, of course, one. Further this holds for multiple trials. Specifically:

$$[P(\alpha<y) + P(\alpha>y)]^n = 1$$

Doing the normal binomial expansion yields:

$$P(\alpha<y)^n + n\,P(a<y)^{n-1}P(\alpha>y) + \ldots\ldots + P(\alpha>y)^n = 1$$

where the first term represents the case where $a$ is always less than y which was used above. The second term contains some of the desired cases which have been omitted. An example of this is one cycle with $a$ just above y and three cycles with very small $a$. However, for the lifetime analysis, these terms can not be included directly in this form because using the term $P(\alpha>y)$ would imply running the density at infinity for one cycle.

## THE TRINOMIAL METHOD

This led to the idea of using the trinomial method. In this case there are two parameters y and z, such that a can be less than (or equal to) y, greater than y but less than (or equal) z or greater than z. Expanding:

$$[P(\alpha<y) + P(y<\alpha<z) + P(\alpha>z)]^n = 1)$$

gives, for n=4,

$$P(\alpha<y)^4 + 4P(\alpha<y)^3 P(y<\alpha<z) +)$$

$$4P(\alpha<y)^3 P(\alpha>z) + 6P(\alpha<y)^2 P(y<\alpha<z)^2 + 12 P(\alpha<y)^2 P(y<a<z) P(\alpha>z)$$

$$.. + 4P(\alpha>z)^3 P(y<\alpha<z) + P(\alpha>z)^4 = 1$$

If the term $P(\alpha<y)$ represents a low density case (L) and $P(y<\alpha<z)$ represents a mid-density case (M) then the first term of the expansion represents four cycles of L and the second term the four permutations of one M and three L's. The basic trinomial method sets the sum of these two terms equal to the desired probability. This gives the relationship between y and z so a single parameter search can be done to maximize the lifetime for a given starting altitude. Preliminary analysis showed that there was little difference between having the M cycle first followed by 3 L cycles compared to having the M cycle later. The standard trinomial method is shown in Figure 1, where the area under the lines at Z and Y represents the probability.

## THE EXTENDED TRINOMIAL METHOD

As the preliminary results showed, the above made a dramatic improvement. However to get even better results, the remaining terms were considered. The extended trinomial method includes portions of these and thus precludes even more of the unneeded conservatism. Perhaps this is best illustrated by considering the table below and Figure 1:

Table 2
PROBABILITY CONTENT ACROSS TERMS

| Term | Full Probab | Cuml Full | Success Prob | Cuml Success |
|---|---|---|---|---|
| 4 L | 66.195 % | 66.195 % | 66.195 % | 66.195 % |
| M, 3L | 25.606 % | 91.801 % | 25.606 % | 91.801 % |
| H, 3L | 3.162 % | 94.963 % | 1.643 % | 93.444 % |
| 2M, 2L | 3.714 % | 98.677 % | 2,195 % | 95.639 % |
| H, M, 2L | 0.917 % | 99.59470 | 0.099 % | 95.73896 |

A y and z are chosen in the regular manner such that they represent success for both the 4L and M, 3L cases. The area above z and below y represents a potential added probability of 3.162%. Note that there is an ultimate upper limit for z where re-entry occurs in 11 years irrespective of having z approach negative infinity (zero density) for the remaining 33 years. After a low level of optimization, the best value of $z^*$ is chosen along with a corresponding $y^*$ to add more probability (cases of success) to the total. Specifically, it is the value of $4P(\alpha<y^*)^3 P(z^*>\alpha>z)$, the shaded area. In this case the added value was 1.643%. Similarly, intermediate values can be found to add portions of the probabilities of the other terms.

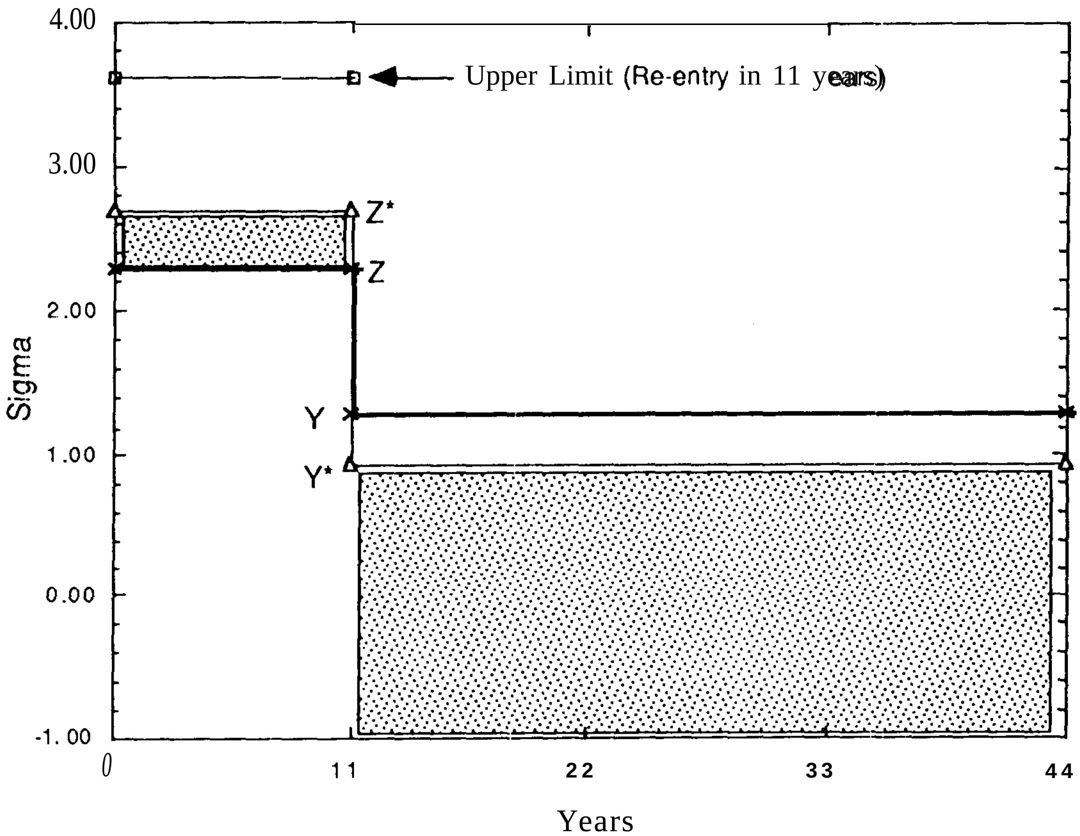


Figure 1 Graphical Representation of the Extended Trinomial Method

The optimization process used would continue to pick other y and z pairs and extensions until the maximum total probability was achieved. In each case the complete success of the first two terms was maintained. An alternative would permit some lack of success in the first two to allow a gain in the other terms. However, a heuristic argument can be made that the ultimate benefit from this "multi-nomial" method is no greater than the trinomial method with extensions. Further, the fact that the benefit gained from each successive term is decreasing dictates that the trinomial plus extensions results are satisfactory for practical purposes.

## RESULTS

Results were obtained in stages, as requirements were changed or tightened and methods were improved. The overall requirements for planetary protection were a 99% probability of success for the first 20 years after launch and a 95% probability for a 50-year time period. The derived requirement for this study was what altitude the satellite needed to be placed after the end of the active mission so it remained in orbit with a certain probability which could be combined with other factors to meet the overall requirements. Although the analysis to meet the 20-year requirement is important, it will be omitted from this paper since the 50-year requirement was not only more stringent but also created the need for the trinomial method.

The original mission plan called for a relay orbit to be maintained for three years after the previous periods of approximately one year of traveling to Mars and obtaining the proper orbit and a two-year mapping mission. Thus, there were 44 years left for the orbit to decay in the 50-year time frame. This was equivalent to four solar cycles and thus was very amenable to the trinomial method. Later the relay phase was reduced to only 6 months so there was an additional two and a half years of decay to be considered. This was done by extending the higher (medium) variation from 11 to 13.5 years. Although this removes the purity of the mathematics of the probability, it can heuristically argued that this extension is a good approximation which errs on the side of conservatism. The extension resulted in the need for 8 km higher orbit (Table 3).

Another complication involved the use of averaged equations of the program POLOP to propagate mean elements. Due to the speed of this type of propagation, it was used for all the iterative runs used to obtain final values. However, then the POHOP program which propagates normal osculating values was used to check the results. It was expected that the use of osculating elements would result in greater orbital decay because the oscillatory nature would imply that the increase in the drag during the lower portions of the orbit would be greater than the decrease during the higher portions because of the exponential relationship between density and height (see below). However, just the opposite effect was found, using POHOP resulted in lower drag and permitted a 4 km lower initial orbit (Table 3). Investigation found this to be a constant increase in the orbital radius due to the higher degree gravitational terms, an interesting orbital mechanics effect in its own right.

Finally, consideration has to be also given to the other parameters of the drag model. For the variation of the atmospheric density with respect to height, a simple exponential model:

$$p = \rho_o \exp(-(h-h_o)/H))$$

was used. The reference height, $h_o$, was chosen to be the Mapping Orbit altitude (378. 1 km) and a scale height, H, equal to 46 km was confirmed to be a good number for altitudes above 200 km where all but the last several months of the orbit occur. However, the best value to use for the reference density was less certain. A re-analysis of the original data determined that the preliminary value of 3.5 x 10”*4 $kg/m^3$ was too conservative and a value of 1.75 x $10^{-14}$ was adopted. This resulted in a 95.7% probability of success from a 427 km altitude. The significant effect of $\rho_o$ changes (Table 3) prompted an independent study (Bougher[2]) which suggested a value of 1.2 x $10^{-14}$. However, to maintain a somewhat consistent level of uncertain y a final value of 1.5 x $10^{-14}$ was chosen to include *some* of the uncertainty of $\rho_o$. With this choice the final probability value was 97.8% (Figure 3) which easily combined with the other contributing factors from other phases to satisfy the 95% requirement.

Table 3
RESULTS FOR VARIOUS TIME PERIODS, SOFTWARE AND REFERENCE DENSITIES
WITH A P= 95.7% OF SUCCESS

| | $\rho_o$ = 3.5 x $10^{-14}$ $kg/m^3$ | $\rho_o$ = 1.75x $10^{-14}$ $kg/m^3$ |
|---|---|---|
| POLOP T= 44 years | 455 km | 423 km |
| POLOP T = 46.5 years | 463 km | 431 km |
| POHOP T =44 years | 451 km | 419 km |
| POHOP T = 46.5 years | 459 km | 427 km |

The other drag parameters used were properties of the MGS satellite in its final configuration. The conservative values: 655 kg, 17 $m^2$ and 2.12 were used for the mass, area and coefficient of ($C_d$) respectively.

## OTHER APPLICATIONS

The trinomial method can be applied to the problem of determining the minimum viable population in ecology. This was studied by setting up a simple example case where the population of one generation was related to the previous generation by: $N_{t+1} = R N_t$ where R is a normal distributed variable with mean equal to unity. The population needed to have a 95%

probability of having 2 individuals left after 10 new generations. Table 4 gives the values using the binomial and trinomial methods for three choices of standard deviation for R. Note that the R* is analogous to the density at y sigma in the above binomial terms while $R_H$ and $R_L$ correspond to the low

and medium densities respectively, in the trinomial terms. Figure 2 shows the linear trend between the needed initial populations and the variance of R. These results indicate a significant reduction in the initial populations needed when the trinomial method is used. Although the suitability of the model would have to be carefully studied before applying it to a real-life situation, this result suggests that common methods of calculating the probability of extinction may be too conservative.

Table 4
MINIMUM INITIAL POPULATIONS USING BOTH METHODS

| Std Dev (σ) of R | Binomial | | Trinomial | | | Compare |
|---|---|---|---|---|---|---|
| | R* | $N_0$* | $R_H$ | $R_L$ | NO | NO*/ NO |
| 0.1 | 0.743 | 39 | 0.8175 | 0.6880 | 18 | 2 |
| 0.2 | 0.486 | 2721 | 0.6324 | 0.3930 | 315 | 9 |
| 0.3 | 0.229 | 5,046,340 | 0.4357 | 0.1384 | 25539 | 198 |

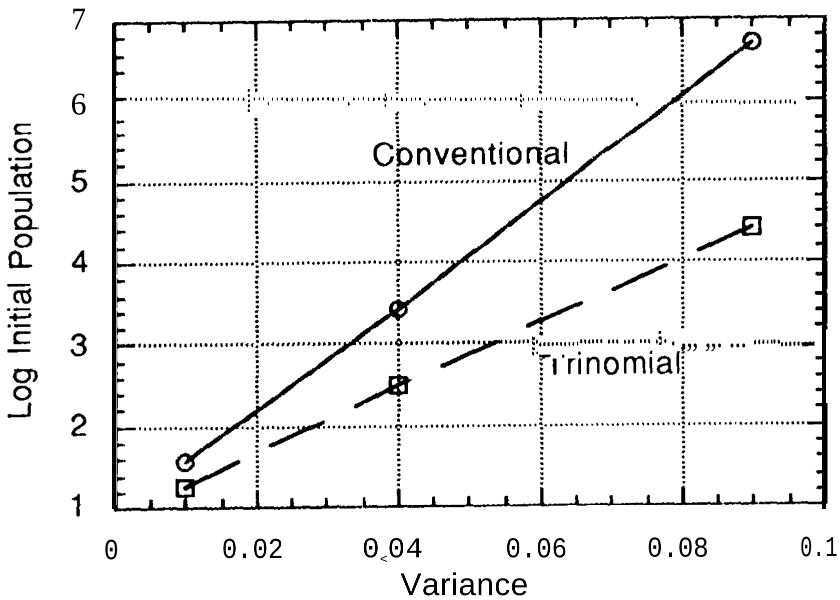


Figure 2 Initial Population vs. Variance of Reproduction Rate

## DISCUSSION OF GENERAL RESULTS AND FUTURE STUDIES

The results of trial and error of the trinomial and extended trinomial method appear to be good and better approximations to the true answer. Although the subsequent behavior after a specific starting altitude represents an ensemble of possible outcomes, each starting value has a corresponding range of final values which map to the probability spectrum. Conversely, the state of zero altitude after 50 years has a corresponding range of starting values. Further, for each starting altitude there exists a single value of density offset for all the cycles which connects it to this final value, call it the "mean offset". The probability value associated with the single cycle. distribution (the uninomial value) is lower than the (tri or) true value. The difference between the two values increases with increasing n because of the tendency of the mean offset to be closer to zero with a larger number of cycles. This is the same as the "variance of the sample mean" decreasing with sample size, as discussed in regular statistics textbooks. Meanwhile the probability from the binomial method is decreasing as the nth power of the uninomial method and thus even more rapidly deviating away from the true answer.

The above paragraph is a rather heuristic method of describing the essence of the trinomial method. Putting the explanation in more formal term will be attempted in later investigations. Another potentially useful exercise would be a Monte Carlo simulation of a large number of random densities, though the best way would probably be to integrate backwards from a zero final condition to find the distribution of initial conditions. Unfortunately, it would probably take too large of a number of runs to achieve the accuracy of the trinomial method but it might be easy to confirm that it is closer than the binomial method.

## CONCLUSIONS

Raising the MGS satellite to an altitude of 427 km after the mission will be completed was accepted by the project management. The trinomial method proved essential because the higher orbits determined by traditional methods incurred impossible requirements on the mission's propellant budget. Specifically, for the same 97.8% probability of success, the binomial method implies that a raise in altitude to 450 km would have been needed. Besides all future Mars orbiters, this method was shown to have a general application to the worst-case analysis of other multiple independent events.

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